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Constitutive Equations in Cosserat Elasticity

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SUMMARY
A new theory for the constitutive equations in Cosserat elasticity is proposed. It is based on the assumption that the rotation vector depending on the displacement vector should be coupled with a rotation vector independent of the displacement vector. This eliminates the indeterminancies in stress and couple-stress encountered earlier.

1. Introduction

A theory of deformation of solid bodies was first indicated by Voigt [1] in the year 1887, but remained unnoticed till the year 1909 when the Cosserat brothers [2] took up the topic and made substantive contributions to it. There was again a gap of forty-four years, and in 1953 Oshima [3] applied this idea to the deformation of granular media. By using this theory Paria [4] attempted to solve the problem of propagation of Love waves in granular media in 1960. From this year, a host of literature followed. They are collected in the historical order in papers of Paria [5]. The medium to which this theory is applied is now-a-days called the Cosserat medium.

The essence of the theory is to introduce the idea of the couple-stress, i.e. the moment per unit area over a specified surface, in addition to the previously prevalent idea of stress, i.e. force per unit area over this surface.

An immediate consequence of the consideration of the couple-stress is that the stress dyadic is found to be not necessarily symmetric. Corresponding to two mechanical entities, namely, the stress dyadic and the couple-stress dyadic, there are two sets of field equations (of motion or equilibrium), and the connecting link between them is maintained by the anti-symmetric part of the non-symmetric stress dyadic. They form the field equations in Cosserat media and have been re-established by many subsequent authors [3, 6, 7, 8].

The next step in the theory is to obtain relations between the stress and couple-stress on the one hand and some kinematical quantities on the other hand. Such relations are called the constitutive equations for the medium.

In a elastic body where the couple-stress is neglected, the constitutive equations are relations between stress and strain dyadics. In fact, intuition suggests that the stress (i.e., force) causes changes in displacement, and we actually establish relations between stress and displacement gradient. Thus the ultimate kinematical quantity in this case is the displacement vector, with the help of which the stress-strain relations are obtained.

Now, if the couple-stress is introduced a corresponding kinematical quantity may be necessary. From intuition, it appears again that a moment causes changes in rotation and hence the couple-stress may be connected with rotation gradients.

In mechanics of continuous media, the rotation vector is defined as half the curl of the displacement vector. Many authors [6, 7, 8, 9] have therefore considered this rotation vector as the second kinematical quantity corresponding to the couple-stress. But it has led to the difficulties that the anti-symmetric part of the stress dyadic as well as the isotropic part of the couple-stress dyadic remain indeterminate. These indeterminacies are perhaps due to the fact that the rotation vector, defined above, is not independent but depends on the displacement vector.

It has therefore been suggested that [10] the rotation vector may be treated as an independent kinematical entity. The physical justification of this hypothesis is that an element of a body can have a rotational motion even without having a translational motion. Now, by considering the displacement and rotational vectors as distinct entities, it is possible to establish constitutive equations which eliminate the indeterminacies mentioned above. But this again introduces certain elements of artificialities in the final definitions of the required kinematical entities, and also in obtaining known results as particular cases [10].

There is therefore a third possibility of generalizing the definition of the rotation vector to include both the independent element and that arising from the displacement vector. Such approach may be found in earlier works of Oshima [3] and Paria [4] in connection with granular media.

The aim of the present paper is to use this generalized definition of rotation to establish constitutive equations in Cosserat elasticity. It is found that there occur no indeterminacies, and the known results may be deduced directly. The field equations can be expressed in terms of the displacement and the independent rotation vectors. The approach is through the strain energy function. In section 2, we review the field equations in dyadic form. Strain energy in general is considered in section 3. In section 4, we justify our approach. In section 5, constitutive equations and field equations for isotropic Cosserat media are considered.

2. The Field Equations

We consider the motion of the material contained within a volume V bounded by the surface S with outward normal vector $\boldsymbol{v}$. Across S there act the traction vector $\boldsymbol{\tau}_v$ and the couple-stress vector $\boldsymbol{\mu}_v$, and within V there act the body-force vector $\boldsymbol{f}$ and body-couple vector $\boldsymbol{c}$. (We shall use bold-face characters to indicate a vector, and a bar to indicate a dyadic. [11])

If $\bar{\tau}$ is the stress dyadic and $\bar{\mu}$ is the couple-stress dyadic, then,

$$\boldsymbol{\tau}_v = \boldsymbol{v}\cdot\bar{\tau}\,, \qquad \boldsymbol{\mu}_v = \boldsymbol{v}\cdot\bar{\mu} \tag{2.1}$$

where the dot denotes the scalar product.

Now, by the principle of balance of momentum we obtain the stress equations of motion

$$\nabla\cdot\bar{\tau} + \rho\boldsymbol{f} = \rho\,\frac{d\boldsymbol{v}}{dt}\,, \tag{2.2}$$

where ρ is the mass per unit volume, d/dt is the material time-derivative and $\boldsymbol{v}$ is the material velocity vector.

Again, by the the principle of balance of moment of momentum, we get the couple-stress equation

$$\nabla\cdot\bar{\mu} + \bar{I}\dot{\times}\bar{\tau} + \rho\boldsymbol{c} = 0\,, \tag{2.3}$$

where $\bar{I} = \nabla\boldsymbol{r}$ and $\boldsymbol{r}$ is the position vector of a point.

An alternative form of (2.3) is

$$\nabla\cdot\bar{\mu} + \bar{\tau}_\times + \rho\boldsymbol{c} = 0\,. \tag{2.4}$$

The equation (2.3) or (2.4) is called the Cosserat equation. Equations (2.2) and (2.3) are the field equations for the Cosserat medium.

From (2.4) it follows that if both $\bar{\mu}$ and $\boldsymbol{c}$ vanish then $\bar{\tau}_\times$ is zero, i.e. the anti-symmetric part of stress $\bar{\tau}$ is zero. The stress is then symmetric. But the converse is not true, i.e. if the stress is symmetric and hence $\bar{\tau}_\times$ is zero, the equation (2.4) does not necessarily vanish identically. Moreover, if any one of $\bar{\mu}$ and $\boldsymbol{c}$ is non-zero, the stress is non-symmetric.

3. Strain Energy

The rotation vector $\boldsymbol{\omega}$ depending upon the displacement $\boldsymbol{u}$ is defined by

$$\boldsymbol{\omega} = \tfrac{1}{2}\nabla\times\boldsymbol{u}\,. \tag{3.1}$$

Let ξ denote the rotation vector independent of the displacement. Then the total rotation vector $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = \boldsymbol{\omega} + \xi \,. \tag{3.2}$$

If W be the strain energy of mass per unit volume, the total strain energy U is

$$U = \int_V W dV \,. \tag{3.3}$$

Since the body is in motion, it has also got kinetic energy. We do not consider here other types of energy.

Now, by the principle of energy, the rate of change of total energy is equal to the rate of work done by external forces. This gives

$$\frac{d}{dt}\int_V \tfrac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{v})\rho\, dV + \frac{d}{dt}\int_V W dV$$
$$= \int_S (\boldsymbol{\tau}_\nu\cdot\boldsymbol{v})dS + \int_S \left(\boldsymbol{\mu}_\nu\cdot\frac{d\boldsymbol{q}}{dt}\right)dS + \int_V (\boldsymbol{f}\cdot\boldsymbol{v})\rho\, dV + \int_V \left(\boldsymbol{c}\cdot\frac{d\boldsymbol{q}}{dt}\right)\rho\, d\varphi \,. \tag{3.4}$$

We transform the surface integrals into volume integrals by the use of divergence theorem in dyadics.

Thus, we get

$$\int_V \frac{dW}{dt}dV = \int_V \left\{\nabla\cdot\bar{\tau} + \rho\boldsymbol{f} - \rho\frac{d\boldsymbol{v}}{dt}\right\}\cdot\boldsymbol{v}\, dV$$
$$+ \int_V (\bar{\tau} : \nabla\boldsymbol{v})dV + \int_V \left[(\nabla\cdot\bar{\mu} + \rho\boldsymbol{c})\cdot\frac{d\boldsymbol{q}}{dt}\right]dV$$
$$+ \int_V \left[\bar{\mu} : \nabla\frac{d\boldsymbol{q}}{dt}\right]dV \,.$$

Using (2.2) and (2.4), and remembering that the volume V is arbitrary, we obtain

$$\frac{dW}{dt} = \bar{\tau} : \nabla\boldsymbol{v} + \bar{\mu} : \nabla\frac{d\boldsymbol{q}}{dt} - (\bar{\tau}_\times)\cdot\frac{d\boldsymbol{q}}{dt} \tag{3.5}$$

Let us write

$$\bar{\tau} = \bar{\tau}^S + \bar{\tau}^A \tag{3.6}$$

where $\bar{\tau}^S$ is the symmetric part and $\bar{\tau}^A$ is the anti-symmetric part of $\bar{\tau}$. For the symmetric part

$$\bar{\tau}^S : \nabla\boldsymbol{v} = \bar{\tau}^S : \boldsymbol{v}\nabla$$

so that

$$\bar{\tau}^S : \nabla\boldsymbol{v} = \bar{\tau}^S : \tfrac{1}{2}(\nabla\boldsymbol{v} + \boldsymbol{v}\nabla) = \bar{\tau}^S : \frac{d\bar{e}}{dt} \tag{3.7}$$

where $\bar{e}$ is the strain dyadic and we have used $\boldsymbol{v} = d\boldsymbol{u}/dt$ for small strain.

For the anti-symmetric part

$$\bar{\tau}^A : \nabla\boldsymbol{v} = -\bar{\tau}^A : \boldsymbol{v}\nabla$$

so that

$$\bar{\tau}^A : \nabla\boldsymbol{v} = \bar{\tau}^A : \tfrac{1}{2}(\nabla\boldsymbol{v} - \boldsymbol{v}\nabla) = \bar{\tau}^A : \frac{d\bar{\omega}}{dt} \tag{3.8}$$

where $\bar{\omega}$ is the rotation dyadic.

From the definitions, it may be verified that

$$\bar{\tau}^A : \frac{d\bar{\omega}}{dt} = (\bar{\tau}_\times) \cdot \frac{d\boldsymbol{\omega}}{dt} \quad . \tag{3.9}$$

Using (3.6), (3.7), (3.8) and (3.9) in (3.5) we obtain

$$\frac{dW}{dt} = \bar{\tau}^s : \frac{d\bar{e}}{dt} + \bar{\mu} : \frac{d\bar{p}}{dt} - \left(\bar{\tau}_\times \cdot \frac{d\xi}{dt}\right) \tag{3.10}$$

where the dyadic $\bar{p}$ is defined by

$$\bar{p} = \nabla \boldsymbol{q} \tag{3.11}$$

Relation (3.10) implies

$$dW = \tau^S_{ij} de_{ij} - \tau^A_i d\xi_i + \mu_{ij} dp_{ij} \tag{3.12}$$

where p_{ij} are components of the dyadic $\bar{p}$ and we have put

$$2\tau^A_{23} = \tau^A_1, \quad 2\tau^A_{31} = \tau^A_2, \quad 2\tau^A_{12} = \tau^A_3 \, . \tag{3.13}$$

From (3.12)

$$\tau^S_{ij} = \frac{\partial W}{\partial e_{ij}}, \quad \tau^A_i = -\frac{\partial W}{\partial \xi_i},$$

$$\mu_{ij} = \frac{\partial W}{\partial p_{ij}} \, . \tag{3.14}$$

These relations determine the stress and couple-stress relations completely, if the form of W is known.

4. Absence of Independent Rotation

If the independent rotation ξ is not considered, the term involving $\bar{\tau}_\times$ in (3.10) is absent. This implies that the anti-symmetric part of the stress dyadic remains indeterminate. Moreover, $\bar{p}$ simplies to

$$\bar{p} = \nabla \boldsymbol{\omega} \, . \tag{4.1}$$

We put

$$\bar{\mu} = \bar{\mu}^0 + \bar{\mu}^D \tag{4.2}$$

where

$$\bar{\mu}^0 = \mu^0 \bar{I} \, , \quad \mu^0 = \tfrac{1}{3}\mu_{ii} \tag{4.3}$$

so that $\bar{\mu}^0$ and $\bar{\mu}^D$ are the isotropic and deviatoric parts of $\bar{\mu}$. We also note the identity

$$\bar{I} : \bar{p} = \bar{I} : \nabla \boldsymbol{\omega} = I : \tfrac{1}{2} \nabla (\nabla \times \boldsymbol{u}) = 0 \, . \tag{4.4}$$

Using (4.2), (4.3) and (4.4), we obtain

$$\bar{\mu} : \frac{d\bar{p}}{dt} = \bar{\mu}^D : \frac{d\bar{p}}{dt} \, . \tag{4.5}$$

By virtue of (4.5), the relation (3.10) becomes independent of $\bar{\mu}^0$. In other words, the isotropic part of $\bar{\mu}$ remains indeterminate.

The indeterminacies of the anti-symmetric part of the stress dyadic and the isotropic part of the couple-stress dyadic, in the absence of the independent rotation, were already encountered by other authors [8, 9].

On the other hand, if we take into account the independent rotation ξ but neglect the dependent rotation ω, as done by Palmov, [10], the indetermincies are no doubt removed, but the definition of the strain dyadic will have to be modified artificially.

We therefore conclude that both types of rotation must be considered [3, 4].

5. Elastically Isotropic Cosserat Solid

From (3.14) it is seen that the strain energy W depends upon the symmetric strain dyadic $\bar{e}$, the non-symmetric torsion-flexure dyadic $\bar{p}$ and the independent rotation vector ξ. If we consider the strains to be small, we may retain only the second order terms in the expansion of W, neglecting higher order terms. The first order terms do not occur if the undeformed state is taken to be the initial state. Moreover, if the medium is elastically isotropic, the invariants corresponding to $\bar{e}$, $\bar{p}$ and ξ may be used. Thus we get

$$\begin{aligned} W = {} & \tfrac{1}{2}(\lambda+2\mu)(e_{11}+e_{22}+e_{33})^2 - 2\mu(e_{11}e_{22}+e_{22}e_{33}+e_{33}e_{11} \\ & -e_{12}e_{21}-e_{23}e_{32}-e_{31}e_{13}) + \tfrac{1}{2}(\alpha_1+2\alpha_2)(p_{11}+p_{22}+p_{33})^2 \\ & -2\alpha_2[p_{11}p_{22}+p_{22}p_{33}+p_{33}p_{11}-\tfrac{1}{4}(p_{12}+p_{21})^2-\tfrac{1}{4}(p_{23}+p_{32})^2-\tfrac{1}{4}(p_{31}+p_{13})^2] \\ & +2\alpha_3[\tfrac{1}{4}(p_{12}-p_{21})^2+\tfrac{1}{4}(p_{23}-p_{32})^2+\tfrac{1}{4}(p_{31}-p_{13})^2] - \tfrac{1}{2}\alpha_4(\xi_1^2+\xi_2^2+\xi_3^2)\,. \end{aligned} \tag{5.1}$$

On the right-hand side of the above relation the first and second terms correspond to the first and second invariants of $\bar{e}$; the third and fourth terms correspond to the first and second invariants of the symmetric part of $\bar{p}$ and the fifth terms corresponds to the invariant of the anti-symmetric part of $\bar{p}$; the last term corresponds to the invariant magnitude of the vectorξ.

It can easily be verified that

$$\tau_{ij}^S = \frac{\partial W}{\partial e_{ij}} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{5.2}$$

where δ_{ij} is the Kronecker delta. Also we have

$$\mu_{11} = \frac{\partial W}{\partial p_{11}} = \alpha_1(p_{11}+p_{22}+p_{33}) + 2\alpha_2 p_{11}\,,$$

$$\mu_{12} = \frac{\partial W}{\partial p_{12}} = \alpha_2(p_{12}+p_{21}) + \alpha_3(p_{12}-p_{21})\,,$$

$$\lambda_{21} = \frac{\partial W}{\partial p_{21}} = \alpha_2(p_{12}+p_{21}) - \alpha_3(p_{12}-p_{21})\,. \tag{5.3}$$

Similar results for μ_{22}, μ_{23}, μ_{32}, ... may be obtained. We introduce the symmetric and anti-symmetric parts of $\bar{\mu}_{ij}$ as

$$\mu_{ij} = \mu_{ij}^S + \mu_{ij}^A$$

where

$$\mu_{ij}^S = \tfrac{1}{2}(\mu_{ij}+\mu_{ji})\,,$$
$$\mu_{ij}^A = \tfrac{1}{2}(\mu_{ij}-\mu_{ji})\,.$$

The results (5.3) then can be written as

$$\mu_{ij}^S = \alpha_1 p_{kk}\delta_{ij} + 2\alpha_2 p_{ij}^S \tag{5.4}$$

$$\mu_{ij}^A = 2\alpha_3 p_{ij}^A \tag{5.5}$$

where p_{ij}^S and p_{ij}^A are the symmetric and anti-symmetric parts of p_{ij}. We also have

$$\tau_i^A = -\frac{\partial W}{\partial \xi_i} = \alpha_4 \xi_i\,. \tag{5.6}$$

The equations (5.2), (5.4), (5.5), and (5.6) are the constitutive equations. There are thus six elastic constants λ, μ, α_1, α_2, α_3 and α_4. The first two are Lamé's constants. Comparing (5.2) and (5.4), we find that the role of α_1 and α_2 are similar to λ and μ respectively. The coefficients α_3 and α_4 are proportional factors for the anti-symmetric parts of couple-stress and stress respectively, as seen from (5.5) and (5.6). Using these constitutive equations, we express the field equation (2.2) and (2.4) as

$$(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})+\mu\nabla^2\boldsymbol{u}+\alpha_4\nabla\times\boldsymbol{\xi}+\rho\boldsymbol{f}=\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \tag{5.7}$$

$$\alpha_1\nabla(\nabla\cdot\boldsymbol{\xi})+(\alpha_2+\alpha_3)\nabla^2\{\tfrac{1}{2}\nabla\times\boldsymbol{u}+\boldsymbol{\xi}\}$$
$$+(\alpha_2-\alpha_3)\nabla\{\nabla\cdot(\tfrac{1}{2}\nabla\times\boldsymbol{u}+\boldsymbol{\xi})\}+\alpha_4\boldsymbol{\xi}+\rho\boldsymbol{c}=0. \tag{5.8}$$

Equations (5.7) and (5.8) are to be solved with appropriate initial and boundary conditions. Illustrations of this theory will be given in subsequent papers.

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